ON THE UNCERTAINTY OF IDENTIFICATION OF CIVIL ENGINEERING STRUCTURES USING ARMA MODELS

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ABSTRACT

In this paper the uncertainties of modal parameters estimated using ARMA models for identification of civil engineering structures are investigated. How to initialize the predictor part of a Gauss-Newton optimization algorithm is put in focus. A backward-forecasting procedure for initialization of the predictor is proposed. This procedure is compared with a standard prediction error method optimization algorithm in a simulation study. It is found that the uncertainties can be reduced by a proper selection of the initial conditions for the predictor.

NOMENCLATURE

$\{\theta\}$	Vector containing ARMA parameters
$\{\varphi\}$	Regression vector
$\{\psi\}$	Gradient filter vector
$\{W\}$	Available measurements vector
$\{\dot{V}\}$	Gradient of loss function
$[H]$	Hessian matrix
$y(t)$	Measured output
$\hat{y}(t)$	Predicted output
$e(t)$	Zero-mean Gaussian white noise
$w(t)$	Auxilliary sequence
$\varepsilon(t)$	Prediction error
μ	Mean value, bisection factor.
β	RMS measure
ζ_j	Damping ratio of the jth mode
f_j	Natural eigen-frequency of the jth mode
T	Sampling period
V	Loss function
q	Forward shift operator
q^{-1}	Backward shift operator
$A(q^{-1})$	Auto Regressive polynomial
$C(q^{-1})$	Moving Average polynomial
a_j	jth Auto Regressive parameter
c_j	jth Moving Average parameter
n_a	Number of Auto Regressive parameters
n_c	Number of Moving Average parameters
t_s	Starting time
N	Number of samples
SNR	Signal-to-noise ratio
CV	Coefficient of variation
$E[\,]$	Expected value
i	$\sqrt{-1}$

1. INTRODUCTION

For several years much research on identification of linear civil engineering structures using an Auto Regressive Moving Average (ARMA) model has been performed, see e.g. Pandit et al. [1]. Several authors have shown that the ARMA model is able to give resonable modal parameter estimates and prediction of the response. However, it is often neglected that identification using ARMA models is a statistical method which allows not only the extraction of the modal parameters from a given measured output; but also estimation of their statistical errors as a measure of the uncertainty, see e.g. Jensen et al. [2]. The quantification of the uncertainties of the modal parameters is especially important if they are used in the field of e.g. vibrational-based inspection. In this case it is important to have an estimate of the uncertainties of the modal parameters, because only significant changes of the modal parameters are of interest, see e.g. Rytter [3], Brincker et al. [4] and Brincker et al. [5]. It is well known that the uncertainties of the modal parameter estimates are dependent upon how the data have been sampled, i.e. the selection of sampling period T, see e.g. D'Emilia et al. [6] and Yao et al. [7]. Much research on the selection of an optimal sampling period that minimizes the uncertainties of the modal parameters has been performed, see e.g. Kirkegaard [8], Ljung [9], Lee et al. [10] and Yao et al. [7]. The general assumption in these cases is that the estimator has asymptotically unbiased and efficient statistical properties, i.e. the estimator attains the Cramer-Rao lower bound of variance. This assumption is only valid if the amount of data is infinite.

This paper concerns the problem that occurs when modal parameters are estimated using an ARMA model with only a limited amount of data. Especially the initialization of the predictor part of the identification procedure will be covered, because this can have significant effect on the estimator properties.

It will be shown how a proper initialization procedure can reduce the uncertainties. Two different implementations of a nonlinear least-square PEM (Prediction Error Method) algorithm will be compared by a simulation study. The algorithm used as reference is the MATLAB [11] routine ARMAX.M which is based on the Gauss-Newton optimization algorithm, see Ljung [12]. This routine can also work without external input, i.e. be used for ARMA models. The performance of this algorithm will be compared with one based on a backward-forecasting method which originally was developed for ARMAX models, where the external input is unknown backwards in time, see Knudsen [13]. This algorithm has proved to work very well on ARMA models as well. In the following the MATLAB routine will be referred to as MAT and the backward-forecasting routine as BAC.

Sections 2 and 3 deal with the foundation of the ARMA model and the PEM method while section 4 concerns the backward-forecasting method. In section 5 an example based on a simulation study is given.

2. ARMA MODEL

If an ARMA($2n$,$2n$-1) model is used for a stationary Gaussian white noise excited linear n-degree of freedom system it can be shown that the covariance of the response due to the ARMA model and that of the white noise excited structure will be identical, see e.g. Kozin et al. [14]. Given a measured response $y(t)$ the ARMA(n_a,n_c) model is defined as

$$\begin{aligned} y(t) + a_1\, y(t-1) + \dots + a_{n_a}\, y(t-n_a) = \\ e(t) + c_1\, e(t-1) + \dots + c_{n_c}\, e(t-n_c) \end{aligned} \tag{1}$$

where $y(t)$ is obtained by filtering the Gaussian white noise $e(t)$ through the filter described by the Auto Regressive polynomial, consisting of n_a parameters a_i, and the Moving Average polynomial, consisting of n_c parameters c_i. By introducing the following polynomials in the backward shift operator q^{-1}, defined as $q^{-j}\, y(t) = y(t-j)$

$$\begin{aligned} A(q^{-1}) = 1 + a_1\, q^{-1} + \dots + a_{n_a}\, q^{-n_a} \\ C(q^{-1}) = 1 + c_1\, q^{-1} + \dots + c_{n_c}\, q^{-n_c} \end{aligned} \tag{2}$$

eq.(1) can be written in a more compact form as

$$y(t) = \frac{C(q^{-1})}{A(q^{-1})}\, e(t) \tag{3}$$

The roots of $A(q^{-1})$ are the poles of the model whereas the roots of $C(q^{-1})$ are the zeroes. Assuming that the model is stable the poles are in complex conjugated pairs. The relationship between the poles p_j and the modal parameters is given by

$$p_j = e^{2\pi f_j T(-\zeta_j + i\sqrt{1-\zeta_j^2})} \qquad j = 1 \dots n_a \tag{4}$$

where f_j and ζ_j are the natural eigen-frequency and damping ratio of the jth mode. T is the sampling period. It is seen that each complex conjugated pair of poles corresponds to a simple-damped oscillator, see Safak [15].

Setting $\lambda_j = \ln(p_j)$ the modal parameters are obtained from the following equations

$$f_j = \frac{|\lambda_j|}{2\pi T}\,, \qquad \zeta_j = \frac{-Re(\lambda_j)}{|\lambda_j|} \tag{5}$$

where || denotes the modulus and Re the real part of the complex number λ_j.

3. OPTIMIZATION PROCEDURE

The parameters of the ARMA model are estimated by minimizing a quadratic error function V also denoted the loss function. Introducing the vector $\{\theta\}$ consisting of the ARMA parameters

$$\{\theta\} = \{a_1, \dots a_{n_a}, c_1, \dots c_{n_c}\}^T \tag{6}$$

$V(\theta)$ is defined as

$$V(\theta) = \frac{1}{2}\frac{1}{N-t_s+1}\sum_{t=t_s}^{N}\varepsilon(t,\theta)^2$$
$$= \frac{1}{2}\frac{1}{N-t_s+1}\sum_{t=t_s}^{N}(y(t) - \hat{y}(t,\theta))^2 \qquad (7)$$

where $\varepsilon(t,\theta)$ is the prediction error, $\hat{y}(t,\theta)$ is the predicted response and $t_s = \max(n_a,n_c)+1$. The optimal one-step predictor is given by Ljung [9]

$$\hat{y}(t,\theta) = \{\varphi\}^T\{\theta\} \qquad (8)$$

where $\{\varphi\}$ is defined as

$$\{\varphi\} = \{-y(t-1),\ldots,-y(t-n_a),\varepsilon(t-1),\ldots,\varepsilon(t-n_c)\}^T \qquad (9)$$

Eq. (8) can be obtained from eq. (3) and (6) by adding the noise term. To calculate the right-hand side of eq. (8) measurements of $y(t)$ from time t-1 and back to the infinite past are necessary. In this case, i.e. the stationary case, $\varepsilon(t,\theta)$ equals the Gaussian white noise $e(t)$ and the predictor is really optimal in the least square sense. To calculate a parameter estimate a numerical minimization method must be chosen. The method which will be used is based on the Gauss-Newton algorithm, defined as

$$\{\theta\}_{k+1} = \{\theta\}_k - \mu[H(\theta_k)]^{-1}\{\dot{V}(\theta_k)\} \qquad (10)$$

where μ is a bisection factor and k is the iteration number. The Hessian matrix $[H(\theta_k)]$ and gradient of the loss function $\{\dot{V}(\theta_k)\}$ are defined as

$$\{\dot{V}(\theta_k)\} = -\frac{1}{N-t_s+1}\sum_{t=t_s}^{N}\{\psi(t,\theta_k)\}\varepsilon(t,\theta_k)$$
$$[H(\theta_k)] = -\frac{1}{N-t_s+1}\sum_{t=t_s}^{N}\{\psi(t,\theta_k)\}\{\psi(t,\theta_k)\}^T \qquad (11)$$

and the regression filter $\{\psi(t,\theta_k)\}$ as

$$\{\psi(t,\theta_k)\} = \frac{1}{C(q^{-1},\theta_k)}\{\varphi(t,\theta_k)\} \qquad (12)$$

The main difference between the two routines is the procedure for initialization of $\varepsilon(t,\theta)$ for $t=t_s$-1 to t_s-n_c in eq. (9). The MAT routine uses a direct start procedure and sets these missing values to zero, i.e. the direct start can be interpreted as the unconditional expectation for these missing values. The BAC routine, on the other hand, uses a conditional expectation for these missing noise values, see Knudsen [13]. This problem concerning initialization of the predictor has been recognised for many years, see Ljung [18]. Especially, a system with a high order $C(q)$ polynomial having weakly damped zeroes close to 1 will give a transient, see e.g. Box et al. [16], Saric et al. [17] and Knudsen [13]. This behaviour will introduce bias and increase the uncertainties of the modal parameters.

To ensure global convergence a good initial estimate of $\{\varphi\}$ must be provided. Both algorithms use the same procedure for establishing the initial estimate. First a high-order AR model is applied to the response $y(t)$. The prediction error $\varepsilon(t)$ of this model is used as external input in an ARX model. The estimated parameters of this model will then be the initial estimate,see Ljung [9] and Ljung [12]. In Ljung [9] it is shown that this will lead to global convergence for the ARMA model optimization procedure.

4. BACKWARD-FORECASTING

In the BAC routine the calculation of the missing initial noise values of the predictor is based on the following conditional expectation

$$E[\{e(t_s-1),\ldots,e(t_s-n_c)\}|\{W\}] = \{e_c(t_s-1),\ldots,e_c(t_s-n_c)\} \qquad (13)$$

given available measurements $\{W\}$. Using this conditional expectation, the transient will disappear, see Knudsen [13]. The index c is used to distinguish this conditional expectation from the one step predictor. To calculate the conditional expectation given by eq. (13) an auxiliary sequence $w(t)$ is introduced

$$w(t) = A(q^{-1})y(t) \qquad (14)$$

This sequence can be calculated for $t=t_s$ to N. Combining eq. (3) with eq. (14) it follows that $w(t)$ is an MA(n_c) process with the forward model defined as

$$w(t) = C(q^{-1})e(t) \qquad (15)$$

and the backward model defined as

$$w(t) = C(q)e^{b}(t) \qquad (16)$$

where the superscript b stands for backwards. Notice that $e(t)$ and $e^b(t)$ are two different white noise sequences. Also notice that $C(q)$ is now a polynomial in the forward shift operator q, defined as $q^j\, y(t) = y(t+j)$.

The development of the backward-forecasting method is based on eq. (15) and (16), and therefore it is sufficient to define the available measurements as the known part of $w(t)$, i.e.

$$\{W\} = \{w(t_s),\dots,w(N)\} \qquad (17)$$

Taking conditional expectation on both sides of eq. (16) yields

$$w_c(t) = C(q)e_c^{b}(t) \qquad (18)$$

It follows from eq. (17) that $w_c(t) = w(t)$, for $t = t_s$ to N. Thus $e_c^b(t)$ can be calculated backwards for $t=N$ to t_s by

$$e_c^{b}(t) = \frac{1}{C(q)}w_c(t) \qquad (19)$$

Starting this filter from $t=N$ requires the initial conditions $\{ e_c^b(N+t), \dots , e_c^b(N+n_c) \}$. These are set to zero and the resulting transient is assumed to have faded out before t_s is reached. Because $e_c^b(t)$ is a zero-mean Gaussian white noise the conditional expected value $E[e_c^b(t)|\{W\}]$ is equal to zero, i.e. $e_c^b(t) = 0$ for $t < t_s$. Using this and eq. (18) it is now possible to predict $w_c^b(t)$ for $t < t_s$.

Taking the conditional expectation on both sides of eq. (15) and seperating $e_c(t)$, yields

$$e_c(t) = \frac{1}{C(q^{-1})}w_c(t) \qquad (20)$$

Because $w_c(t)= 0$ for $t < t_s-n_c$ it follows from eq. (20) that $e_c(t) = 0$ for $t < t_s-n_c$. Inserting $w_c(t)$ for $t = t_s-n_c$ to t_s-1 eq. (20) the initial conditions $\{ e_c(t_s-1), \dots , e_c(t_s-n_c) \}$ can be calculated. Using these initial conditions it is straightforward to calculate the loss function V in eq. (7) for a specified value of $\{\theta\}$.

In Knudsen [13] it is shown that the initial values of the regression filter in eq. (12), for $t = t_s-n_c$ to t_s-1 can be obtained as

$$\{\psi(t)\} = -\frac{de_c(t)}{d\{\theta\}} \qquad (21)$$

These initial values can be calculated by differentiating eq. (14) - (16) and eq. (18) - (20) with respect to $\{\theta\}$, see Knudsen [13].

5. EXAMPLE

The significance of different initialization procedures is highly dependent upon the chosen model order. A first order ARMA model will not show any particular differences, see Knudsen [15]. In this section an ARMA(10,9) model will be used. This model is the covariance equivalent to a linear 5-degree of freedom uncoupled system excited by a Gaussian white noise. The natural eigenfrequencies f_j and the corresponding damping ratios ζ_j are seen in table 1. In table 2 the parameters of the corresponding ARMA model are listed. In fig. 1 the poles and zeroes of the ARMA model are plotted. The zeroes are seen to be close to the complex unit circle indicating that a large transient can be expected. The system has been simulated using a sampling frequency equal to 75 Hz.

The example consists of four runs. In table 2 the number of simulations N_{sim} and the amount of data N of each run are shown. In the table the signal-to-noise ratio SNR of each run is also shown. The SNR is defined as the ratio between the standard deviation of the simulated response and the added Gaussian white noise.

For comparison, two RMS measures $\{\beta_f\}$ and $\{\beta_\zeta\}$, defined as

$$\{\beta_f\} = \sqrt{\frac{1}{N_{sim}}\sum_{i=1}^{N_{sim}}(\{\hat{f}\}_i - \{f\})^2}$$
$$\{\beta_\zeta\} = \sqrt{\frac{1}{N_{sim}}\sum_{i=1}^{N_{sim}}(\{\hat{\zeta}\}_i - \{\zeta\})^2} \qquad (22)$$

are introduced. $\{\beta_f\}$ and $\{\beta_\zeta\}$ are the *RMS*-values of the differences between estimated and model values of the natural eigen-frequencies and damping ratios respectively.

The standard deviation of the prediction error for runs 1 and 4 is shown in figs. 1 and 2 at each time step as examples. It should be noted how the standard deviations in the transient parts are reduced using the BAC routine. In tables 4 to 7 the mean value, the coefficient of variation and the *RMS* of the estimated modal parameters are listed. It is seen that in every run the results obtained from using the BAC routine regarding the natural eigen-frequencies are superior to the results obtained from the MAT routine. The results regarding the damping ratios are on the other hand not so good for either of the routines.

6. CONCLUSION

In this paper it is shown that the uncertainties and the bias of the modal parameters, applying an ARMA model to a small amount of data, can be reduced by a proper initialization of the predictor part of the optimization algorithm. One way of making this reduction is using a backward-forecasting procedure to predict the un-known initial values. The results have been verified by a simulation study comparing an ordinary optimization algorithm with one using the backward-forecasting approach.

ACKNOWLEDGEMENT

The Danish Technical Research Council is gratefully acknowledged.

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Mode	f_i [Hz]	ζ_i [%]
1	2	0.1
2	4	0.5
3	8	0.1
4	16	5
5	32	10

Table 1. The modal parameters of the linear structural system.

	$A(q^{-1})$	$C(q^{-1})$
1	-4.493	-3.239
2	8.581	4.599
3	-8.584	-3.601
4	3.808	1.656
5	1.085	-0.547
6	-2.010	0.326
7	-0.368	-0.039
8	2.311	-0.316
9	-1.832	0.231
10	0.509	

Table 2. The ARMA model that corresponds to the linear system.

Run	SNR %	N	N_{sim}
1	5	200	500
2	20	200	500
3	5	1000	250
4	20	1000	250

Table 3. The four simulation runs dependence on SNR, N and N_{sim}.

	MAT			BAC		
	μ	CV	β	μ	CV	β
f_1	2.06	0.053	0.122	2.05	0.052	0.120
f_2	3.91	0.100	0.400	3.91	0.087	0.352
f_3	8.04	0.041	0.328	8.04	0.038	0.306
f_4	16.55	0.115	1.974	16.31	0.109	1.800
f_5	31.55	0.093	2.956	31.12	0.095	3.080
ζ_1	0.052	0.957	0.071	0.056	1.010	0.079
ζ_2	0.108	1.132	0.160	0.112	1.270	0.180
ζ_3	0.030	2.545	0.083	0.064	2.750	0.185
ζ_4	0.114	1.165	0.148	0.109	1.270	0.151
ζ_5	0.041	1.690	0.091	0.051	1.267	0.081

Table 4. Results of run 1. Mean value μ, coefficient of variation CV and RMS measure β of the modal parameters f_j and ζ_j.

	MAT			BAC		
	μ	CV	β	μ	CV	β
f_1	2.05	0.052	0.120	2.05	0.048	0.110
f_2	3.80	0.117	0.483	3.91	0.097	0.387
f_3	8.07	0.057	0.464	8.05	0.058	0.467
f_4	17.27	0.126	2.492	16.88	0.125	2.295
f_5	30.95	0.099	3.228	30.80	0.096	3.178
ζ_1	0.069	1.011	0.097	0.067	1.033	0.096
ζ_2	0.197	1.117	0.215	0.156	1.061	0.223
ζ_3	0.050	2.066	0.113	0.060	2.541	0.162
ζ_4	0.115	1.392	0.171	0.131	1.273	0.186
ζ_5	0.040	1.427	0.083	0.045	1.428	0.084

Table 5. Results of run 2. Mean value μ, coefficient of variation CV and RMS measure β of the modal parameters f_j and ζ_j.

	MAT			BAC		
	μ	CV	β	μ	CV	β
f_1	2.03	0.033	0.075	2.01	0.018	0.037
f_2	4.08	0.081	0.338	4.04	0.059	0.241
f_3	7.97	0.059	0.470	7.99	0.023	0.185
f_4	16.63	0.119	2.064	16.52	0.088	1.690
f_5	31.83	0.107	3.579	30.56	0.094	2.975
ζ_1	0.044	1.189	0.068	0.027	1.271	0.042
ζ_2	0.118	1.394	0.199	0.053	1.837	0.108
ζ_3	0.018	5.783	0.105	0.100	2.337	0.253
ζ_4	0.118	1.304	0.168	0.081	1.356	0.113
ζ_5	0.032	1.266	0.079	0.031	1.183	0.077

Table 6. Results of run 3. Mean value μ, coefficient of variation CV and RMS measure β of the modal parameters f_j and ζ_j.

	MAT			BAC		
	μ	CV	β	μ	CV	β
f_1	2.05	0.044	0.104	2.04	0.038	0.084
f_2	4.10	0.094	0.382	4.08	0.051	0.222
f_3	8.08	0.048	0.203	8.02	0.025	0.394
f_4	17.51	0.137	2.405	16.53	0.108	2.300
f_5	31.29	0.106	3.524	30.58	0.097	3.095
ζ_1	0.057	1.174	0.088	0.057	1.163	0.087
ζ_2	0.151	1.478	0.258	0.059	1.563	0.105
ζ_3	0.024	4.687	0.112	0.091	2.660	0.258
ζ_4	0.135	1.611	0.232	0.327	1.036	0.435
ζ_5	0.046	1.414	0.084	0.046	1.834	0.100

Table 7. Results of run 4. Mean value μ, coefficient of variation CV and RMS measure β of the modal parameters f_j and ζ_j.

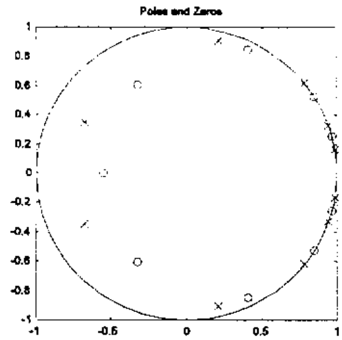

Fig 1. Poles (x) and zeroes (o) of system.

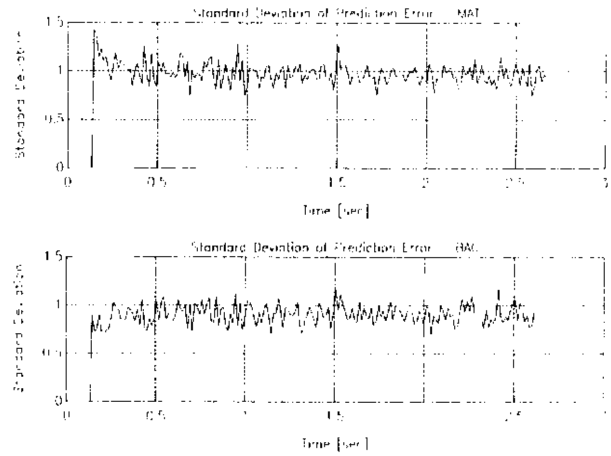

Fig 2. Standard deviation of prediction error using the simulations of Run 1.

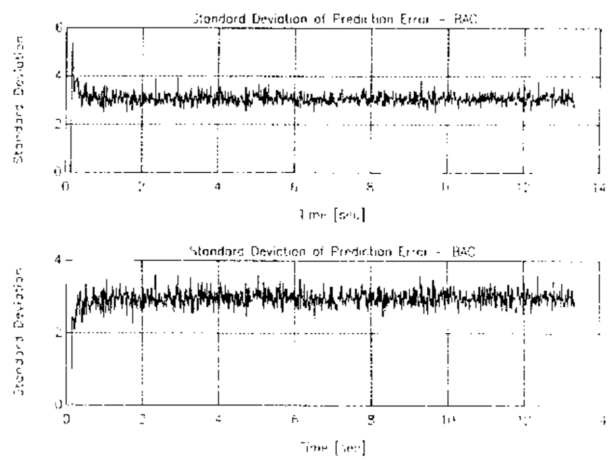

Fig 3. Standard deviation of prediction error using the simulations of Run 4.